INF 240 - Exercise problems - 3

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Exercise 1. *Write down Cayley tables for the additive and multiplicative operation in* $(\mathbb{F}_7, +, \cdot)$. *In addition, create tables of additive and multiplicative inverses which lists the inverse of each element of* $\mathbb{F}_7$.

+	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

Table 1: Addition table for $\mathbb{F}_7$

$\cdot$	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

Table 2: Multiplication table for $\mathbb{F}_7$

x	$-x$	x^{-1}
0		
1		
2		
3		
4		
5		
6		
7		

Table 3: Inverse table for $\mathbb{F}_7$

Exercise 2. *Consider the ring $\mathbb{F}_7[x]$ of polynomials in indeterminate x over the finite field $(\mathbb{F}_7, +, \cdot)$.*

1. *Write down the vectors of coefficients corresponding to*
$$p_1(x) = x^5 + 3x^4 + 6x^2 + 2x + 1$$
and
$$p_2(x) = 6x^4 + 3x^3 + x^2 + x + 5;$$
2. *What are the degrees of $p_1(x)$ and $p_2(x)$?*
3. *Is $p_1(x)$, resp. $p_2(x)$ a monic polynomial?*
4. *Compute $p_1(x) + p_2(x)$, $p_1(x) \cdot p_2(x)$, $-p_1(x)$ and $-p_2(x)$;*
5. *Divide $p_1(x)$ by $p_2(x)$ with remainder, i.e. find $q(x)$ and $r(x)$ satisfying $p_1(x) = q(x)p_2(x) + r(x)$, with $\deg r(x) < \deg p_2(x)$.*

Exercise 3. *Consider the set of 3×3 matrices over the real numbers. Recall that addition of matrices is performed component-wise, i.e.*

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}.$$

Matrix multiplication is a bit more involved; we have

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix},$$

where $c_{i,j} = \sum_{k=1}^{3} a_{i,k} b_{k,j}$. For example, $c_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}$. Recall also the zero matrix

$$Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the identity matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} C = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}.$$

1. *The zero matrix clearly has the property that $Z + M = M + Z$ for any matrix M. Show that the identity matrix has the same property with respect to matrix multiplication, i.e. $IM = MI = M$ for any matrix M.*
2. *Compute the sums $A + B$, $A + C$, $B + C$.*

3. *Compute the products AB, BA, AC, CA, BC, CB.*

4. *What are the additive inverses of A, B and C?*

Exercise 4. *Consider the matrices A, B, and C defined in the previous exercise but over $\mathbb{F}_5$ instead of the real numbers. Repeat the same operations as in the previous exercise:*

Compute the sums $A+B$, $A+C$, $B+C$.

Compute the products AB, BA, AC, CA, BC, CB.

What are the additive inverses of A, B and C.

Exercise 5. *(Lidl & Niederreiter 1.15)*

Recall from the lecture that if R is a commutative ring of characteristic p, then for any $a, b \in R$ we have $(a+b)^{p^n} = a^{p^n} + b^{p^n}$. Show that

$$(a_1 + a_2 + a_3 + \cdots + a_k)^{p^n} = a_1^{p^n} + a_2^{p^n} + a_3^{p^n} + \cdots + a_k^{p^k}$$

for any $a_1, a_2, a_3, \ldots, a_k \in R$, where $k \in \mathbb{N}$.

Hint: Use Theorem 1.46 from Lidl & Niederreiter and proceed by induction on k.