INF 240 - Exercise problems - 2

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Exercise 1. *(Lidl & Niederreiter 1.2)*
Let $(G, \cdot)$ *be a multiplicative group, and let* $\emptyset \neq H \subseteq G$.

1. *Show that* H *is a subgroup of* G *if and only if*

$$a, b \in H \implies ab^{-1} \in H \tag{1}$$

for all $a, b \in G$.

2. *Show that if* H *is finite, then the condition in* (1) *can be replaced by*

$$a, b \in H \implies ab \in H$$

for all $a, b \in G$.

Exercise 2. *(Lidl & Niederreiter 1.4)*
For a natural number m, *Euler's function* $\varphi(m)$ *is defined to be the number of integers* k *such that* $1 \leq k \leq m$ *and* $\gcd(k, m) = 1$. *Show that for any* $m, n, s, p \in \mathbb{N}$ *with* p *prime:*

1. $\varphi(p^s) = p^s \left(1 - \frac{1}{p}\right)$;
2. *if* m *and* n *are prime numbers, then* $\gcd(m, n) = 1$, *then* $\varphi(mn) = \varphi(m)\varphi(n)$.

Exercise 3. *(Lidl & Niederreiter 1.7)*
Let $(R, +, \cdot)$ *be a ring. Show that*

$$(-a)(-b) = ab$$

for all $a, b \in R$.

Exercise 4. *Consider the additive groups* $(\mathbb{Z}_8, +)$ *and* $(\mathbb{Z}_{10}, +)$.

1. *Write out the Cayley tables for* $(\mathbb{Z}_8, +)$ *and* $(\mathbb{Z}_{10}, +)$;
2. *Find a normal subgroup* H *of* $(\mathbb{Z}_8, +)$ *and a normal subgroup* N *of* $(\mathbb{Z}_{10}, +)$;
3. *Construct the factor groups* $\mathbb{Z}_8/H$ *and* $\mathbb{Z}_{10}/N$;
4. *Find the index and order of* H *in* $\mathbb{Z}_8$ *and of* N *in* $\mathbb{Z}_{10}$;
5. *Find generators of* $\mathbb{Z}_8$, $\mathbb{Z}_{10}$, M, *and* N;
6. *Find the conjugacy classes and the center of* $\mathbb{Z}_8$ *and of* $\mathbb{Z}_{10}$.