Ex 11, for nit007, Kenneth Fossen

Ex 1

consider the boolean function f on 4 variables given by the ANF

$$f(x_1 x_2 x_3 x_4) = 1 + x_2 + x_1 x_3 + x_1 x_2 x_4$$

x_1	x_2	x_3	x_4	f
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	1
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	1
1	1	1	0	1
1	1	1	1	0

$$1 + x_2 + x_1 x_3 + x_1 x_2 x_4$$

1234		1		x_2		$x_1 \cdot x_3$		$x_1 \cdot x_2 \cdot x_4$		
0000	=	1	+	0	+	$0 \cdot 0$	+	$0 \cdot 0 \cdot 0$	=	1
0001	=	1	+	0	+	$0 \cdot 0$	+	$0 \cdot 0 \cdot 1$	=	1
0010	=	1	+	0	+	$0 \cdot 1$	+	$0 \cdot 0 \cdot 0$	=	1
0011	=	1	+	0	+	$0 \cdot 1$	+	$0 \cdot 0 \cdot 1$	=	1
0100	=	1	+	1	+	$0 \cdot 0$	+	$0 \cdot 1 \cdot 0$	=	2 = 0
0101	=	1	+	1	+	$0 \cdot 0$	+	$0 \cdot 1 \cdot 1$	=	0
0110	=	1	+	1	+	$0 \cdot 1$	+	$0 \cdot 1 \cdot 0$	=	0
0111	=	1	+	1	+	$0 \cdot 1$	+	$0 \cdot 1 \cdot 1$	=	0
1000	=	1	+	0	+	$1 \cdot 0$	+	$1 \cdot 0 \cdot 0$	=	1
1001	=	1	+	0	+	$1 \cdot 0$	+	$1 \cdot 0 \cdot 1$	=	1
1010	=	1	+	0	+	$1 \cdot 1$	+	$1 \cdot 0 \cdot 0$	=	0
1011	=	1	+	0	+	$1 \cdot 1$	+	$1 \cdot 0 \cdot 1$	=	0
1100	=	1	+	1	+	$1 \cdot 0$	+	$1 \cdot 1 \cdot 0$	=	0
1101	=	1	+	1	+	$1 \cdot 0$	+	$1 \cdot 1 \cdot 1$	=	1
1110	=	1	+	1	+	$1 \cdot 1$	+	$1 \cdot 1 \cdot 0$	=	1
1111	=	1	+	1	+	$1 \cdot 1$	+	$1 \cdot 1 \cdot 1$	=	0

Ex2 Consider the Boolean function g on 4 variables given by TT find the ANF

$g_6(\;) = 1$

$g_9(\;) = 1$

$g_{12}(\;) = 1$

$g = g_6 + g_9 + g_{12}$

From the TT:

$g_6 = (0,1,0,1) = (x_1+1)x_2(x_3+1)x_4$

$g_9 = (1\,0\,0\,0) = x_1(x_2+1)(x_3+1)(x_4+1)$

$g_{12} = (1\,0\,1\,1) = x_1(x_2+1)x_3x_4$

$g_6 = (x_1+1)x_2(x_3+1)x_4$

$= (x_2x_1 + x_2)(x_4x_3 + x_4)$

$g_6 = x_2x_1x_4x_3 + x_2x_1x_4 + x_2x_4x_3 + x_2x_4$

$= x_1x_2x_3x_4 + x_1x_2x_4 + x_2x_3x_4 + x_2x_4$

$\cancel{0\,1\,0}\;1 \quad \cancel{0\,1\,1} \quad \cancel{1\,0\,1} \quad 1\cdot 1$ ok

$$g_9 = x_1(x_2+1)(x_3+1)(x_4+1)$$

$$= x_1(x_2+1)(x_3x_4+x_3+x_4+1)$$

$$x_1(x_2x_3x_4 + x_2x_3 + x_2x_4 + x_2 + x_3x_4 + x_3 + x_4 + 1)$$

$$= x_1x_2x_3x_4 + x_1x_2x_3 + x_1x_2x_4$$

$$1\ 0\ 0\ 0 \quad 1\ 0\ 0 \quad 1\ 0\ 0$$

$$+ x_1x_2 + x_1x_3x_4 + x_1x_3 + x_1x_4 + x_1$$

$$1\ 0 \quad 1\ 0\ 0 \quad 1\ 0 \quad 1\ 0 \quad 1$$

$$= 1$$

ok

$$g_{12} = x_1(x_2+1)x_3x_4$$

$$= (x_1x_2 + x_1)x_3x_4$$

$$g_{12} = x_1x_2x_3x_4 + x_1x_3x_4$$

$$1\ 0\ 1\ 1 + 1\ 1\ 1 = 1 \quad \text{ok}$$

$g(x_1x_2x_3x_4) = g_6 + g_9 + g_{12}$

$= x_1x_2x_3x_4 + \cancel{x_1x_2x_4} + x_2x_3x_4 + x_2x_4$

$x_1x_2x_3x_4$	$\cancel{x_1x_2x_4}$	$x_2x_3x_4$	x_2x_4
0 1 0 1	0 1 1	1 0 1	1 · 1
1 0 0 0	1 0 0	0 0 0	0 0
1 0 1 1	1 0 1	0 1 1	0 1

$+ \cancel{x_1x_2x_3x_4} + x_1x_2x_3 + \cancel{x_1x_2x_4} + x_1x_2$

$\cancel{x_1x_2x_3x_4}$	$x_1x_2x_3$	$\cancel{x_1x_2x_4}$	x_1x_2
0 1 0 1	0 1 0	0 1 0	0 1
1 0 0 0	1 0 0	1 0 0	1 0
1 0 1 1	1 0 1	1 0 1	1 0

$+ \cancel{x_1x_3x_4} + x_1x_3 + x_1x_4 + x_1$

$\cancel{x_1x_3x_4}$	x_1x_3	x_1x_4	x_1
0 0 1	1 0	1 1	1
1 0 0	1 0	1 0	1
1 1 1	1 · 1	1 · 1	1

$+ \cancel{x_1x_2x_3x_4} + \cancel{x_1x_3x_4}$

$\cancel{x_1x_2x_3x_4}$	$\cancel{x_1x_3x_4}$		
0 1 0 1	0 0 1	$= 3 = 1$	ok
1 0 0 0	1 0 0	$= 1 = 1$	ok
1 0 1 1	1 1 1	$= 5 = 1$	ok

$g(x_1 x_2 x_3 x_4)$ check if g_6 g_9 and g_{12} pass

$= x_1 x_2 x_3 x_4 + x_2 x_3 x_4 + x_2 x_4$

	x_1	x_2	x_3	x_4	x_2	x_3	x_4	x_2	x_4
g_6:	1	0	0	0	0	0	0	0	0
g_9:	1	0	1	1	0	1	1	0	1
g_{12}:	0	1	0	1	1	0	1	1	1
g_4	0	0	1	1	0	1	1	0	1

$+ x_1 x_2 x_3 + x_1 x_2 + x_1 x_3 + x_1 x_4 + x_1$

x_1	x_2	x_3	x_1	x_2	x_1	x_3	x_1	x_4	x_1	
1	0	0	1	0	1	0	1	0	1	= 1
1	0	1	1	0	1	1	1	1	1	= 1
0	1	0	0	1	0	0	0	1	0	= 1
0	0	1	0	0	0	1	0	1	0	= 0

Ok

$$g = x_1 x_2 x_3 x_4 + x_2 x_3 x_4 + x_2 x_4 + x_1 x_2 x_3 + x_1 x_2 + x_1 x_3 + x_1 x_4 + x_1$$

algebraic degree of $g = 4$ $(x_1 x_2 x_3 x_4)$

Ex3 Suppose H is $(3,4)$

$h_1(x_1, x_2\, x_3) = 1 + x_1 + x_1 x_2$

$h_2(x_1, x_2\, x_3) = 1 + x_1 x_2$

$h_3(x_1, x_2\, x_3) = x_1 x_2$

$h_4(x_1, x_2\, x_3) = x_1 + x_1 x_2 x_3$

$$H = (\underbrace{1 + x_1 + x_1 x_2}_{h_1},\ \underbrace{1 + x_1 x_2}_{h_2},\ \underbrace{x_1 x_2}_{h_3},\ \underbrace{x_1 + x_1 x_2 x_3}_{h_4})$$

ANF $H = (1,1,0,0) + (1,1,1,1)x_1 + (1,1,1,1)x_2 + (0,0,0,1)x_3$

ANF Degree $H = 1$

x_1	x_2	x_3	$F(x_1\ x_2\ x_3\ x_4)$ TT			
0	0	0	1	1	0	0
0	0	1	1	1	0	0
0	1	0	1	1	0	0
0	1	1	1	1	0	0
1	0	0	0	1	0	1
1	0	1	0	1	0	1
1	1	0	1	0	1	1
1	1	1	1	0	1	0

Ex 4

Consider (3,2)-fund

$F = (f_1, f_2)$ ∧ $G = (g_1, g_2)$

1. Hamming distance between $f_1 f_2$

distance is $\underline{d(f_1 f_2) = 3}$

				x_i
for inputs	$(0,0,1)$ =	1	0	= 1
	$(0,1,1)$ =	1	0	= 1
	$(1,0,0)$ =	0	1	= 1

$\Sigma x_i = 3$

2. Hamming distance between $g_1\ g_2$

		x_i
$(0,0,1)$ =	$0,1$ =	1
$(1,0,0)$ =	$1,0$ =	1
$(1,1,0)$ =	$0,1$ =	1

$\Sigma x_i = 3$

distance is $\underline{\underline{d(g_1 g_2) = 3}}$

3. Hamming distance between F and G

2? 3?

		F	G
input	(0,1,0) =	1,1	1,1
	(1,1,1) =	1,1	1,1
	(1,0,0) =	0,1	0,1

ok

$d(F,G) = 8-3 = \underline{\underline{5}}$

We only found 3 similar outputs from the input.

Ex5 For the following

(10,10)-functions given in univariate form, compute

algebraic degree
univariate degree

Are any functions linear, affine, quadratic, cubic?

1. $F(x) = x^{22} + x^{17} + \alpha x^6 + x^4 + 1$

univariate degree = 22

$4 = 100 \quad = 1$

$6 = 110 \quad = 2$

$17 = 10001 \quad = 2$

$22 = 10110 \quad = 3$

algebraic degree is 3 = 10110 = 22

function is cubic
since the degree is 3

2. $G(x) = \alpha x^{15} + x^{11} + x^{3}$

univariat degree = 15

$3 = 11 \qquad = 2$

$11 = 1011 \qquad = 3$

$15 = 1111 \qquad = 4$

algebraic degree is $\underline{\underline{4}}$

function is linear

$G(x) = \sum_{i=0}^{n-1} b_i x^{2^i}$

3. $f(x) = x^{256} + \alpha^{17} x^{64} + x^{8} + x^{4} + \alpha^{2} x^{2} + 1$

Univariat degree = 256

$256 = 100000000 = 1$

$64 = 10000000 = 1$

$8 = 1000 = 1$

$4 = 100 = 1$

$2 = 10 = 1$

algebraic degree = 1

function is affine

it adds a constant term and is of degree 1